IFUSP-P- 1246

Quasi-Topological Field Theories in Two Dimensions as Soluble Models

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Abstract

We study a class of lattice field theories in two dimensions that includes gauge theories. Given a two dimensional orientable surface of genus g, the partition function Z is defined for a triangulation consisting of n triangles of area ϵ. The reason these models are called quasi-topological is that Z depends on g, n and ϵ but not on the details of the triangulation. They are also soluble in the sense that the computation of their partition functions can be reduced to a soluble one dimensional problem. We show that the continuum limit is well defined if the model approaches a topological field theory in the zero area limit, i.e., $\epsilon \to 0$ with finite n. We also show that the universality classes of such quasi-topological lattice field theories can be easily classified. Yang-Mills and generalized Yang-Mills theories appear as particular examples of such continuum limits.

1 Introduction

Exactly soluble models in statistical mechanics [1] and field theory are extremely valuable examples where one hopes to learn about the physics of more realistic models for which exact calculations are not available. The Ising model, for instance, has proven to be an incredible source of important ideas, such as duality and finite size scaling [2], that can be applied to much more general situations.

The simplest examples of soluble models are probably the so called lattice topological field theories [3, 4, 5, 6]. Let M be an oriented 2-dimensional compact manifold and T a triangulation of M. For instance, the authors of [4], starting from a quite general ansatz, determined what were the conditions on local Boltzmann weights such that the partition $Z(T)$ was independent of T. In other words, $Z(T)$ was proven to be a topological invariant of M. A large class of models, corresponding to semi-simple associative algebras, was found. The reason we say that these lattice theories are soluble is because to compute $Z(T)$ for a triangulation T with an arbitrary number n of triangles it is enough to take another triangulation T^0 with the minimal number of triangles and compute $Z(T^0)$ explicitly.

Lattice Topological Field Theories (LTFT) are in a sense very simple. They are almost trivial from the point of view of dynamics. Consider for example a cylinder with boundary $S^1 \cup S^1$, and the corresponding evolution operator U (or transfer matrix in the language of statistical systems). It is trivial to show that for a LTFT the operator U is equal to the identity when restricted to physical observables *. Despite their simplicity, topological models represent an attractive class of models. They can be generalized to higher dimensions and still be exactly soluble. The same type of models considered in [4] have been carried out in 3 dimensions [5]. A different approach have been used by the authors of [6] to produce subdivision invariant theories in several dimensions, including four.

There is a large variety of fully dynamical soluble theories in $d = 2$ [1], but in dimensions bigger than 2 this is far from being true. Unfortunately the general situation is that physical models in higher dimensions are either soluble but too simple as LTFT's, or dynamically nontrivial but too hard be be exactly solved, as for example lattice gauge theories in 3 dimensions. It would be desirable to find a class of models interpolating these two extreme situations. We want to look for models that are a little more dynamical than LTFT and still can have its partition function computed. The answer is not known in general, but two dimensional Yang-Mills theories (YM_2) are legitimate examples of such models. It is well known that the partition function of a gauge theory on a 2-manifold M is not a topological invariant. Nevertheless its partition function can be explicitly computed in the continuum and in the lattice [7]. It turns out that the partition function depends not only on the topology of M but also on its area α. Yang-Mills is a deformation of a topological theory in the sense that it reduces to a topological theory in the limit

*However, if instead of a cylinder one has some other manifold interpolating the two circles $S^1 \cup S^1$, U is no longer the identity.

$\alpha \to 0$. This is an example of what can be called a $2d$ quasi-topological field theory. It is well known that YM_2 is a particular deformation of a topological theory known as $2d$ topological BF theory. There are other examples of quasi-topological theories that are deformations of the same BF theory, where the deformation parameter is again the area. They correspond to gauge theories known as generalized YM_2 [8].

In this paper we shall discuss how to construct quasi-topological theories on the lattice. They will include gauge theories as a particular example. Let M_g be an orientable 2d surface with genus g, and $T(g,n)$ a triangulation of M_g consisting of n triangles. For simplicity, we will assume that all triangles have the same area ϵ. To each link in $T(g,n)$ we associate a dynamical variable taking values in a discrete (or even continuous) set I. Then we follow [4] and look for models such that the partition function $Z(T(g,n),\epsilon)$ depends on the topology through g, on the total number n of triangles, and on ϵ but not on the details of the triangulation T. In other words, Z is a function $Z(g,n,\epsilon)$ of the topology, the number of triangles and their size. That will be our definition of a lattice quasi-topological field theory (LQTFT). As we shall see, if the set I of dynamical variables is finite, the partition function can be exactly computed. In any case, an explicit expression for $Z(g,n,\epsilon)$ can always be given. We will show that the continuum limit of a LQTFT is well defined whenever the model is a deformation of a lattice topological theory, i.e, it approaches a topological field theory in the zero area limit. The continuum limit is recovered by taking $n \to \infty$ and $\epsilon \to 0$, while keeping the total area $\alpha = n\epsilon$ fixed.

We start by defining a lattice quasi-topological field theories in Section 2. In Section 3 we compute the partition function in full generality. The dynamics of LQTFT is discussed in Section 4. There we compute the evolution operator U for the case of a cylinder and comment on how to extend the answer to a generic situation. We also determine what are the physical observables and compare with the topological case. In Section 5 we study the continuum limit. Section 6 is dedicated to a simple example that corresponds to the generalized YM_2. Finally on Section 7, we conclude with some remarks. Some relevant results on triangulation of 2d surfaces are given in the Appendix.

2 Quasi-Topological Lattice Theories

The definition of the model is inspired by [4]. Let $T(g,n)$ be a triangulation with n triangles of a two dimensional surface M_g with genus g. For simplicity, we will assume that all triangles have the same area ϵ. A configuration is determined by assigning to each edge of the triangulation a "color" i belonging to a index set I. For gauge theories, I is nothing but the gauge group G. To each triangle of area ϵ, with edges colored by i,j,k, we associate a Boltzmann weight $C_{ijk}(\epsilon)$. We assume that all triangles have the same area ϵ and that $C_{ijk}(\epsilon)$ is invariant by cyclic permutation of the color indexes, i.e.,

$$C_{ijk}(\epsilon) = C_{jki}(\epsilon) = C_{kij}(\epsilon). \tag{2.1}$$

An arbitrary triangulation consists of n triangles glued pairwise along their edges. Therefore it is enough to specify what is the Boltzmann weight associate with two joined triangles. Consider the example in Fig. 1. The corresponding weight is determined by a gluing operator $g^{kl} = g^{lk}$ and is given by

$$C_{ija}(\epsilon)\ g^{ab}\ C_{bkl}(\epsilon), \tag{2.2}$$

where summations on the repeated indexes a and b are understood. One may use g^{ab} to lift indexes and write (2.2) as $C_{ij}{}^{b}(\epsilon)C_{bkl}(\epsilon)$ or $C_{ija}(\epsilon)C^{a}{}_{kl}(\epsilon)$.

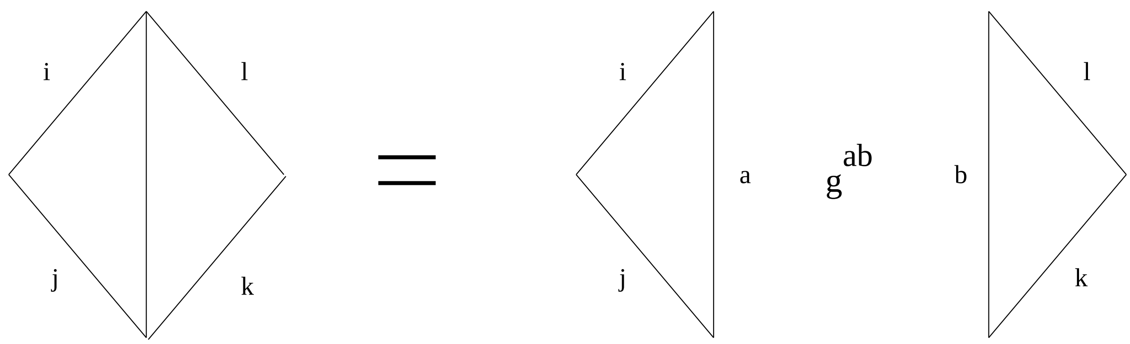


Fig.1. The figure shows how the gluing operator g^{ab} is used to give the weight corresponding to a pair of glued triangles.

It will be convenient to restrict the gluing operator g^{ij} in such way that there exists an inverse g_{ij},

$$g_{ia}g^{aj} = \delta_i^j. \tag{2.3}$$

The partition function for the triangulation $T(g,n)$ is obtained by performing the gluing operation on all pairs $< ab >$ of edges that should be identified in order to build the triangulation. In other words,

$$Z(T(g,n),\epsilon) = \prod_{\Delta\in T}\prod_{<ab>} C_{ijk}(\epsilon)g^{ab}. \tag{2.4}$$

If the weights $C_{ijk}(\epsilon)$ are not restricted, the partition function (2.4) depends on the triangulation and it can be a complicated and fully dynamical theory.

It is convenient to represent a given triangulation $T(g,n)$ by its dual graph $\Gamma(g,n)$. Fig. 2 (a) shows the gluing of triangles in terms of the dual graphs. For an arbitrary graph, such as the one in Fig. 2 (b), one iterates the gluing on its trivalent elementary pieces. The resulting weight associated with $\Gamma(g,n)$ will be a number that depends on the variables $i_1, i_2, \ldots, i_6$ attached to the external legs. The graphs must have double lines in order to encode the same information as the triangulation.

Given two triangulations $T(g,n)$ and $T'(g,m)$, or equivalently the corresponding graphs $\Gamma(g,n)$ and $\Gamma'(g,m)$, of a surface with genus g, it is possible to transform one

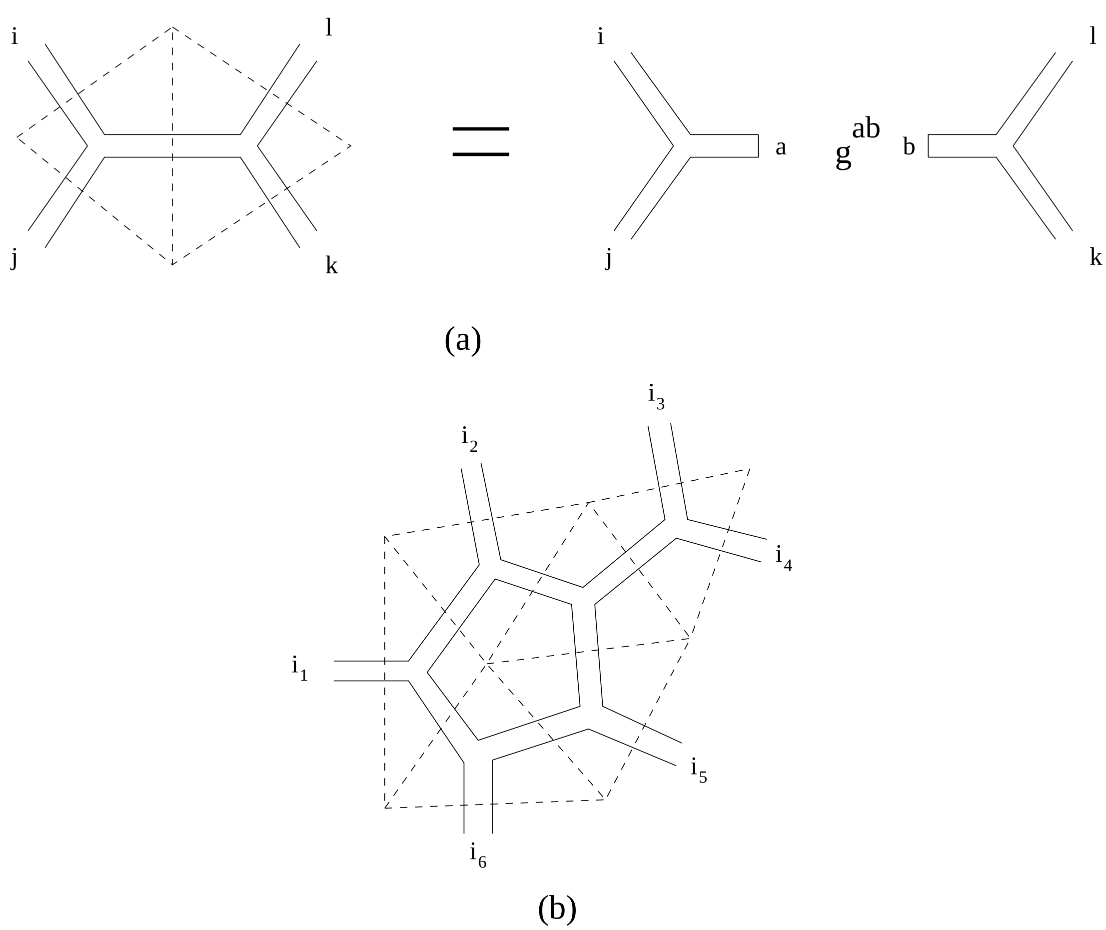
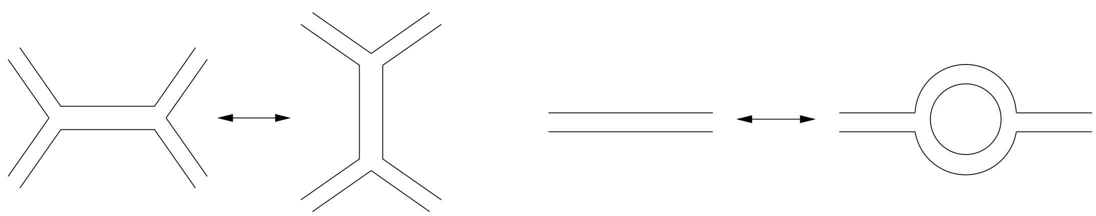


Fig.2. Figure (a) shows the gluing of triangles in terms of the dual graph. Figure (b) is a simple example of a triangulation and its dual graph.

into another by a set of local moves that do not change the topology, namely g. It is well known that two basic moves are needed in order to go from one triangulation to another. We are going to use the so called flip move and bubble move. In terms of the dual graphs, these moves are given in Fig. 3. Note that the flip move preserves the number n of triangles, whereas the bubble move changes it by two.



(a) (b)

Fig.3. The two basic topological moves in terms of the dual graphs. Figure (a) is the flip move and figure (b) is the bubble move.

For a topological theory [4] $C_{ijk}(\epsilon)$ does not depending on ϵ, i. e.,

$$\frac{dC_{ijk}(\epsilon)}{d\epsilon} = 0 \tag{2.5}$$

and it is invariant under both topological moves. Invariance under the flip move implies that

$$C_{ij}{}^{k} C_{kl}{}^{m} = C_{ik}{}^{m} C_{jl}{}^{k}, \tag{2.6}$$

whereas the bubble move is equivalent to

$$C_{iab} C^{ba}{}_{j} = g_{ij}. \tag{2.7}$$

A partition function that is invariant under both moves, can not depend on the triangulation, and therefore it is a topological invariant of the triangulated surface. In other words, Z is a function $Z(g)$ depending only on the genus g of the surface M_g.

A topological theory defined by C_{ijk} has an enormous symmetry. Thanks to this fact, the partition function can be computed. Since Z do not depend on the triangulation, one chooses the minimal triangulation and writes down $Z(g)$ explicitly. Topological models are very special when compared to a generic theory given by (2.4). In general a model defined by (2.4) have little or no symmetry at all. What we are going to do is to consider an intermediate situation where part of the full topological symmetry is not present. That is another reason for the name lattice quasi-topological field theory (LQTFT).

The simplest thing to do is to give up the invariance under one of the two topological moves described before. It will be interesting to have a partition function that depends

on the size of the lattice, so we choose to break the invariance under the bubble move and keep the invariance under the flip move. We also want to allow for variation on the size ϵ of the triangles. The model is defined by a set of local weights $C_{ijk}(\epsilon)$ invariant under the flip move, and partition function given by (2.4). In other words, our class of models satisfy the flip move

$$C_{ij}{}^{k}(\epsilon)C_{kl}{}^{m}(\epsilon) = C_{ik}{}^{m}(\epsilon)C_{jl}{}^{k}(\epsilon) \tag{2.8}$$

for any value of the parameter ϵ.

It may happen that for some critical value $\epsilon = \epsilon_0$, the weights $C_{ijk}(\epsilon)$ also satisfy equation (2.7). At this critical point, the full topological symmetry is restored. As we shall see, if $\epsilon = 0$ is a critical point, the model has a well defined continuum limit.

Let us assume for simplicity that the index set I is a finite set with r elements. Consider a vector space V with bases $\{\phi_1, ..., \phi_r\}$. Then, for each value of the parameter ϵ the numbers $C_{ij}{}^{k}(\epsilon)$ define a one parameter family of product structures in V, namely

$$\phi_i\phi_j := C_{ij}{}^{k}(\epsilon)\phi_k. \tag{2.9}$$

Because of the flip symmetry (2.8) the product $\phi_i\phi_j$ is associative. We may think of $C_{ij}{}^{k}(\epsilon)$ as given a family A_ϵ of algebras on the space of associative algebras defined on V.

Since we are assuming that g^{ij} has an inverse g_{ij} we can define a dual base $\{\phi^i\}$ given by

$$\phi^i = g^{ij}\phi_j. \tag{2.10}$$

For the dual basis, the product is

$$\phi^i\phi^j := C^{ij}{}_{k}(\epsilon)\phi^k. \tag{2.11}$$

The data we need to define a LQTFT is a pair (A_ϵ, g^{ij}) of one parameter family of algebras and a bilinear form. There may be some topological critical point $\epsilon = \epsilon_0$ where equation

$$C_{iab}(\epsilon_0)C^{ba}{}_{j}(\epsilon_0) = g_{ij}$$

is valid. Such point determines a TFT specified by the data (A_{ϵ_0}, g^{ij}).

3 Partition Function

For a triangulation $T(0,n)$ of the sphere, the corresponding graphs $\Gamma(0,n)$ are planar. Let $\Gamma(0,n)$ and $\Gamma'(0,n)$ be two planar graphs representing two different triangulations of S^2 but with the same number n of triangles. It is a well known fact that $\Gamma(0,n)$ and $\Gamma'(0,n)$ can always be connected via a sequence of flip moves [9]. Therefore if $C_{ijk}(\epsilon)$ fulfills equation (2.8) the partition function (2.4) computed for $\Gamma(0,n)$ and $\Gamma'(0,n)$ have to be identical.

Using the same idea of the proof presented in [9] we were able to show that any pair of dual graphs $\Gamma(g,n)$ and $\Gamma'(g,n)$, for arbitrary genus g, can also be connected by a sequence of flip moves. For completeness we give a demonstration of this fact on the Appendix. As a result, our partition function (2.4) depends only on g,n and ϵ, provide that (2.8) is fulfilled. We will write $Z = Z(g,n,\epsilon)$ for this matter. The particular graph $\Gamma(g,n)$ used to compute Z is immaterial.

The result of the Appendix shows that any graph $\Gamma(g,n)$ can be reduced via a sequence of flip moves to the standard graph $\Gamma^0(g,n)$ given on Fig. 4(a).

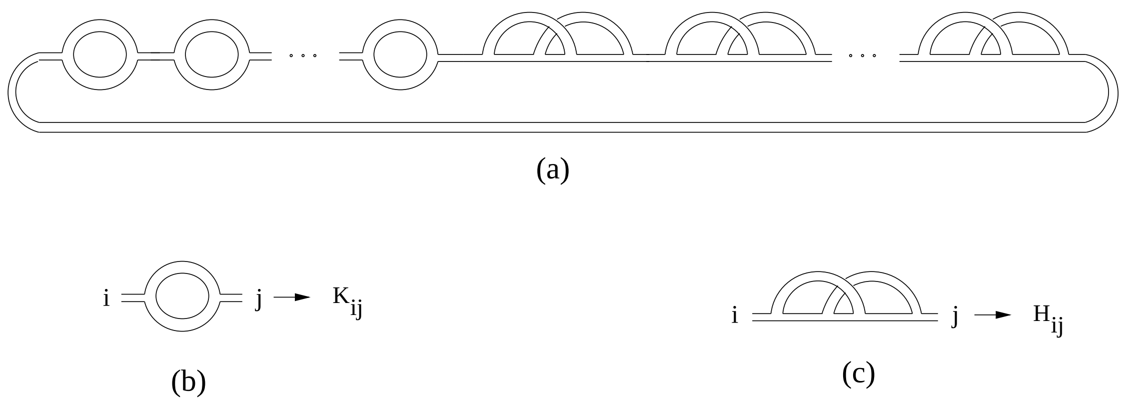


Fig.4. The dual graph corresponding to the standard triangulation of a surface of genus g is given in Figure (a). It can be constructed by repeating the basic blocks shown in Figures (b) and (c) respectively $\frac{n-4g}{2}$ and g times.

The standard graph is obtained by gluing the elementary blocks on Fig. 4(b) and Fig. 4(c). From the gluing rules of last Section follows that the elementary block of Fig. 4 (b) correspond to the number $K_{ij}(\epsilon)$ given by

$$K_{ij}(\epsilon) := C_{iab}(\epsilon)\ C^{ba}{}_{j}(\epsilon) \tag{3.1}$$

where ij are the variables attached to the external legs. It is straightforward to verify that

$$K_{ij}(\epsilon) = K_{ji}(\epsilon). \tag{3.2}$$

Analogously, the elementary block in Fig. 4 (c) gives

$$H_{ij}(\epsilon) := C_{ikl}(\epsilon)\ C^{kmn}(\epsilon)\ C_{m}{}^{pl}(\epsilon)\ C_{npj}(\epsilon) \tag{3.3}$$

with

$$H_{ij}(\epsilon) = H_{ji}(\epsilon). \tag{3.4}$$

If we define matrices $K(\epsilon)$ and $H(\epsilon)$ with matrix elements $[K(\epsilon)]_i{}^j = K_{ik}(\epsilon)g^{kj}$ and $[H(\epsilon)]_i{}^j = H_{ik}(\epsilon)g^{kj}$, then the graph on Fig. 4(a) shows that $Z(g,n,\epsilon)$ can be written as

$$Z(g,n,\epsilon) = Tr\left(K(\epsilon)^{\frac{n-4g}{2}} H(\epsilon)^g\right) \tag{3.5}$$

Equation (3.5) and Fig. 4 (a) show that the computation of the partition function for any 2d lattice surface has been reduced to a one dimensional problem. If the set I of states is a discrete set with r elements, $K(\epsilon)$ and $H(\epsilon)$ are $r \times r$ matrices. In this case, (3.5) can be calculated for an arbitrary g and n. For this note that the algebra of observables generated by $\{\phi_1, ..., \phi_r\}$ has a natural inner product given by

$$< \phi_i, \phi_j >= g_{ij}. \tag{3.6}$$

Using (3.2) and (3.4) one can verify that

$$\begin{aligned} < \phi_i, K_j{}^l(\epsilon)\phi_l > &= < K_i{}^l(\epsilon)\phi_l, \phi_j >; & (3.7)\\ < \phi_i, H_j{}^l(\epsilon)\phi_l > &= < H_i{}^l(\epsilon)\phi_l, \phi_j > . & (3.8)\end{aligned}$$

In other words, $K(\epsilon)$ and $H(\epsilon)$ are self-adjoint with respect to the inner product (3.6). Moreover, we will see next that they also commute

$$K(\epsilon)H(\epsilon) = H(\epsilon)K(\epsilon), \tag{3.9}$$

therefore they can be simultaneously diagonalized. As the trace is unchanged by a coordinate transformation, the partition function can be computed as

$$Z(g,n,\epsilon) = \sum_{l=1}^{r} k_l{}^{\frac{n-4g}{2}}\ h_l{}^g, \tag{3.10}$$

where k_l and h_l are the eigenvalues of $K(\epsilon)$ and $H(\epsilon)$.

We now show that equation (3.9) is fulfilled. This is a direct consequence of the flip symmetry. Consider the graphic representation of $K_i{}^a(\epsilon)H_{aj}(\epsilon)$ on Fig. 5(a). By performing a flip transformation, the leg of the graph marked with 2 can be moved to the position presented on Fig. 5(b). Repeating the same step one can move it further, arriving at Fig. 5(c). Finally, Fig. 5(d) is obtained by repeating the process with leg 1. The interpretation of Fig. 5 (d) in terms of $K_i{}^j(\epsilon)$ and $H_{ij}(\epsilon)$ reads

$$K_i{}^l(\epsilon)H_{lj}(\epsilon) = H_i{}^l(\epsilon)K_{lj}(\epsilon). \tag{3.11}$$

Therefore $K(\epsilon)$ and $H(\epsilon)$ commute.

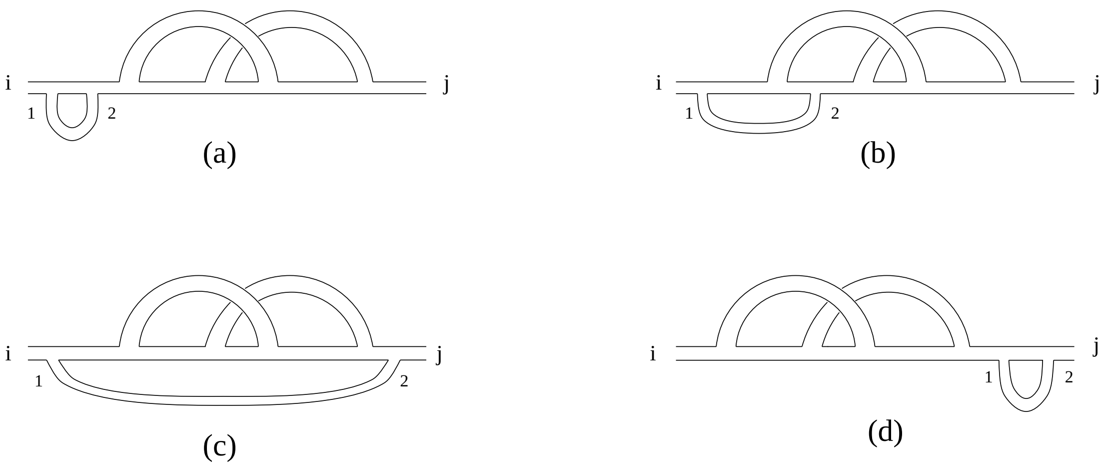


Fig.5. The figure shows equation (3.11).

4 Dynamics and Observables

Consider a cylinder where the configuration at the boundary is fixed. The result of summing over all internal configurations gives us what we call the evolution operator, or in the language of statistical mechanics, the iterated transfer matrix. We will denote the corresponding operator by U. The dynamical aspects of a model is determined by U, since in the continuum U may be written as the exponential of some Hamiltonian. By its definition, a topological theory is such that U is equal to the identity when restricted to the physical observables. Since our models have less symmetry than a LTFT we expect that it must have some dynamics.

Let $T(p_1, p_2, n)$ be a triangulation of a cylinder where n is the number of triangles and p_1, p_2 are the number of edges (or links) on the boundaries σ_1 and σ_2. For each boundary circle σ_1 and σ_2 we choose a starting link and enumerate the edges in a clockwise fashion. An example is shown in Fig. 6. We define the operator $U_{i_1,\ldots,i_{p_1};\; j_1,\ldots,j_{p_2}}$ as the one given by the gluing (summing over) of the internal links according to the rules explained in Section 2, while keeping the boundary configurations on σ_1 and σ_2 fixed as $(i_1, \ldots, i_{p_1})$ and $(j_1, \ldots, j_{p_2})$ respectively. In other words,

$$U_{i_1,\ldots,i_{p_1};\; j_1,\ldots,j_{p_2}} = \prod_{\Delta \in T} \prod_{<ab>} C_{ijk}(\epsilon) g^{ab} \tag{4.1}$$

where $< ab >$ runs over the pairs of glued internal links. What we will call the evolution operator is the matrix

$$U^{j_1,\ldots,j_{p_2}}_{i_1,\ldots,i_{p_1}} := U_{i_1,\ldots,i_{p_1};\; k_1,\ldots,k_{p_2}} g^{k_1 j_1} \ldots\; g^{k_{p_2} j_{p_2}}. \tag{4.2}$$

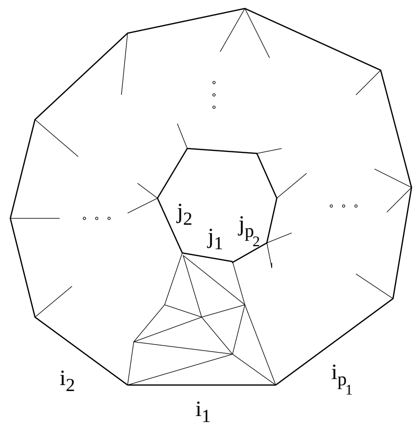


Fig.6. A cylinder with boundary given by two polygons with p_1 and p_2 links. The boundary elements are enumerated in a clockwise fashion.

It is clear from the definition that $U^{j_1,...,j_{p_2}}_{i_1,...,i_{p_1}}$ fulfills the factorization properties of an evolution operator. Consider the splitting of $T(p_1, p_2, n)$ in two cylinders $T_a(p_1, p, n_a)$ and $T_b(p, p_2, n_b)$, $n_a + n_b = n$, then

$$U^{j_1,...,j_{p_2}}_{i_1,...,i_{p_1}}(T) = U^{k_1,...,k_p}_{i_1,...,i_{p_1}}(T_a)U^{j_1,...,j_{p_2}}_{k_1,...,k_p}(T_b). \tag{4.3}$$

We are going to assume for the moment that the set I of variables is equal to $\{1, 2, ..., r\}$. Then, the vector space $V \sim A_\epsilon$ of states associated with a single link is generated by a basis $\{\phi_1, \phi_2, ..., \phi_r\}$. In other words, a generic state ψ is given by $\psi = \psi^i\phi_i$. The space of states $V^{(p_1)}$ corresponding to the boundary component σ_1 with p_1 links is just the tensor product $V^{(p_1)} = V \otimes V \otimes ... \otimes V$ with p_1 factors. At the boundary σ_2 the space of states $V^{(p_2)}$ is defined in the same way. We recall the usual interpretation for U as an linear operator from $V^{(p_1)}$ to $V^{(p_2)}$ given by

$$U(\phi_{i_1} \otimes ... \otimes \phi_{i_{p_1}}) = U^{j_1,...,j_{p_2}}_{i_1,...,i_{p_1}}\phi_{j_1} \otimes ... \otimes \phi_{j_{p_2}}. \tag{4.4}$$

The computation of U follows the same idea as in the calculation of the partition function in Section 3. Given two triangulations $T(p_1, p_2, n)$ and $T'(p_1, p_2, n)$ with the same number of triangles, and the same number of links on the boundary, we were able to show that they can be connected by a sequence of flip moves. Therefore U depends only on the triangulation through the numbers p_1, p_2 and n. In fact any triangulation $T(p_1, p_2, n)$ can be brought to the standard form given on Fig. 7. The proof is analog to the one presented on the Appendix section.

Note that once more the computation has been reduced to a one dimensional problem. It involves the product of the operator $K_i^{\ j}(\epsilon)$ given in (3.1), and a new operator $S_i^{\ j}(\epsilon)$ defined by

$$S_i^{\ j}(\epsilon) := C_{iab}(\epsilon)C^{abj}(\epsilon). \tag{4.5}$$

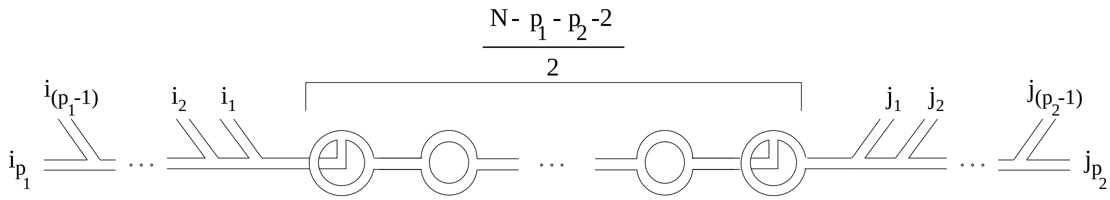


Fig.7. The figure shows the standard graph for a cylinder. The chain of operators $K_i^{\ j}(\epsilon)$ starts and ends at a new operator $S_i^{\ j}(\epsilon)$.

We are going to use the following property of $S_i^{\ j}(\epsilon)$:

$$S_i^{\ m}(\epsilon)C_{mjk}(\epsilon) = S_i^{\ m}(\epsilon)C_{mkj}(\epsilon). \tag{4.6}$$

A diagrammatic proof of (4.6) is given in Fig. 8.

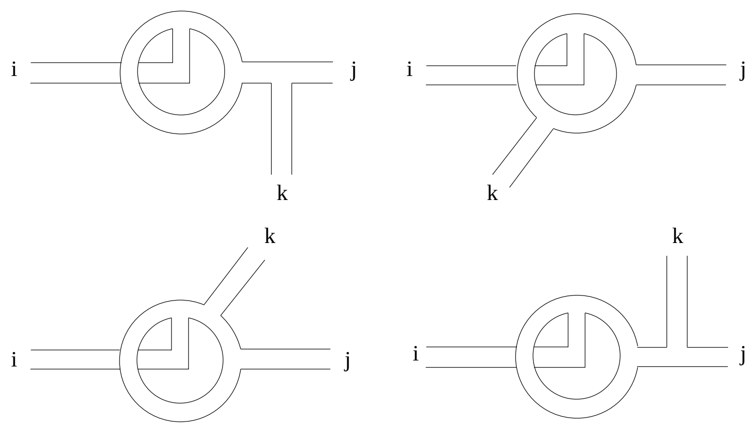


Fig.8. The figure shows a sequence of moves that proofs equation (4.6).

Now let us consider the linear map $S_\epsilon : A_\epsilon \to A_\epsilon$ given by $S_i^j(\epsilon)$. On a generic element $a = a^i\phi_i \in A_\epsilon$ it act as

$$S_\epsilon(a) = a^i S_i^{\ j}(\epsilon)\phi_j \ . \tag{4.7}$$

Using equations (4.6) and (2.1), it is a simple matter to verify that for any $a_1, a_2 \in A_\epsilon$

$$a_1 S_\epsilon(a_2) = S_\epsilon(a_2)a_1 \tag{4.8}$$

and

$$S_\epsilon(a_1a_2) = S_\epsilon(a_2a_1). \tag{4.9}$$

Equation (4.8) shows that $S_\epsilon(a)$ belongs to the center $Z(A_\epsilon)$ of the algebra A_ϵ. In other words, S_ϵ maps the algebra A_ϵ on its center $Z(A_\epsilon)$. The square of S_ϵ can also be computed. One can show that

$$S_i{}^l(\epsilon)S_l{}^j(\epsilon) = K_i{}^l(\epsilon)S_l{}^j(\epsilon) = S_i{}^l(\epsilon)K_l{}^j(\epsilon). \tag{4.10}$$

Note that in the topological case $K(\epsilon)$ is the identity and S_ϵ becomes a projector.

One can see from Fig. 7 that the form of $U_{i_1,\dots,i_{p_1}}^{j_1,\dots,j_{p_2}}(p_1,p_2,n,\epsilon)$ is

$$U_{i_1,\dots,i_{p_1}}^{j_1,\dots,j_{p_2}}(p_1,p_2,n,\epsilon) = [S_\epsilon(\phi_{k_1}\phi_{k_2}...\phi_{k_p})]^a \left(K(\epsilon)^{\frac{n-p_1-p_2-2}{2}}\right)_a^m [S_\epsilon(\phi^{j_{p_2}}\phi^{j_{p_2-1}}...\phi^{j_1})]_m. \tag{4.11}$$

The first fact one should notice about (4.11) is that $U(p_1,p_2,n,\epsilon)$ is not an arbitrary operator, but its matrix elements depend only on $S_\epsilon(\phi_{i_1}...\phi_{i_{p_1}})$ and $S_\epsilon(\phi^{j_{p_2}}\phi^{j_{p_2-1}}...\phi^{j_1})$. Therefore the evolution given by $U(p_1,p_2,n,\epsilon)$ is actually an evolution for the data $S_\epsilon(\phi_{i_1}...\phi_{i_{p_1}})$ belonging to the center $Z(A_\epsilon)$. Furthermore, equation (4.9) implies that S_ϵ is invariant under cyclic permutations

$$S_\epsilon(\phi_{i_1}...\phi_{i_{p_1}}) = S_\epsilon(\phi_{i_{p_1}}\phi_{i_1}...\phi_{i_{p_1-1}}). \tag{4.12}$$

For this reason, it becomes useful to introduce loop variables analogue to the trace of the Wilson loop in gauge theories. Let σ be a loop on the lattice made of an oriented sequence $(1,2,...,p_1)$ of links. Then we define the loop variable $W(\sigma) \in Z(A_\epsilon)$ as

$$W(\sigma) = S_\epsilon(\phi_{i_1}...\phi_{i_{p_1}}). \tag{4.13}$$

Note that $W(\sigma)$ depends on the orientation of σ, but not on its starting point. Analogously, we define its conjugate $\tilde{W}(\sigma) = W(-\sigma)$, where $-\sigma$ is the same loop with reverse orientation, by

$$\tilde{W}(\sigma) = S_\epsilon(\phi^{j_{p_2}}\phi^{j_{p_2-1}}...\phi^{j_1}) \in Z(A_\epsilon) \tag{4.14}$$

The matrix elements of U depend only on $W(\sigma_1)$ and $\tilde{W}(\sigma_2)$, and therefore in analogy with gauge theories one should regard the loop variables as the observables of the theory.

Note that, when restricted to the observables, the evolution U is given by

$$U|_{\text{phy}} = K(\epsilon)^{\frac{n-p_1-p_2-2}{2}}. \tag{4.15}$$

The observables, or loop variables $W(\sigma)$, take values in the set $L(A_\epsilon)$ of elements of the center of the form $S_\epsilon(a)$ for some $a \in A_\epsilon$. A natural question to ask is whether $L(A_\epsilon)$ is the entire $Z(A_\epsilon)$ or just a subspace. The answer will depend on the particular set of weights $C_{ijk}(\epsilon)$. However it is possible to give a sufficient condition such that $L(A_\epsilon) = Z(A_\epsilon)$. Consider an element $z = z^i\phi_i \in Z(A_\epsilon)$. Using the fact that $z^iC_{ijk} = z^iC_{jik}$ one can show that

$$S_i{}^j(\epsilon)z^i = K_i{}^j(\epsilon)z^i. \tag{4.16}$$

Therefore if $K(\epsilon)$ restricted to the center is invertible then $L(A_\epsilon)$ is equal to $Z(A_\epsilon)$. For example this is what happens for the topological case when $K_i{}^j = \delta_i{}^j$.

A Cylinder is topologically equivalent to a sphere with two holes. For this reason U is also called the two point correlator for genus zero. To complete our discussion we should consider the corresponding operator for a surface with g handles and N holes, i.e., the N points correlator for genus g. It is a well known result that it is sufficient to compute the three point correlator Y for genus zero. Any other correlator can be written in terms of Y and U. Consider a sphere with 3 holes representing a cobordism from $S^1 \times S^1$ to S^1. Let $T(p_1, p_2, p_3, n)$ be a triangulation with n triangles and p_i links on the oriented boundary σ_i. It is not difficult to show that analogously to (4.11) we have

$$Y^{k_1,\dots,k_{p_3}}_{i_1,\dots i_{p_1};\ j_1,\dots,j_{p_2}}(p_1, p_2, p_3, n) = [W(\sigma_1)]^a\ [W(\sigma_2)]^b\ C_{ab}{}^l(\epsilon)\ [K(\epsilon)^{\frac{q}{2}}]_l{}^m [\tilde{W}(\sigma_3)]_m, \tag{4.17}$$

where $q = n - p_1 - p_2 - p_3 - 4$.

5 Continuum Limit

The continuum limit is obtained by taking the number n of triangles going to infinity. We will be interested in the scaling situation, when the area ϵ of each triangle becomes smaller but the total area α of the surface remains constant. Therefore ϵ and n are related by

$$\epsilon = \frac{\alpha}{n}. \tag{5.1}$$

At this limit, the partition function will be a function $Z(g, \alpha)$ of the genus g and the area α.

Consider the weights associated with the two triangles of Fig. 9 (a). In the continuum limit ϵ goes to zero and both triangles have zero area. Therefore their weights should be equal. The corresponding diagrams are shown in Fig. 9 (b). It is clear from Fig. 9 (b) that $C_{ijk}(0)$ should satisfy the equation

$$C_{iab}(0) C_j{}^{ba}(0) = g_{ij} \tag{5.2}$$

or, in other words $K_i{}^j(0) = \delta_i^j$. But (5.2) is exactly the condition (2.7) to have a lattice topological field theory. Hence, to have a well defined continuum limit, the point $\epsilon = 0$ has to be a critical point characterized by the algebra A_0. Therefore, to have a well defined continuum limit, the weights $C_{ijk}(\epsilon)$ have to be a deformation of a LTFT. Therefore

$$C_{ijk}(\epsilon) = C^{top}_{ijk} + \epsilon \frac{\partial}{\partial \epsilon} C_{ijk}(0) + \mathcal{O}(\epsilon^2). \tag{5.3}$$

Using (5.2) and (5.3) in the definition of $K_i{}^j$ we have

$$K_i{}^j(\epsilon) = \delta_i^j + 2\epsilon B_i{}^j + \mathcal{O}(\epsilon^2) \tag{5.4}$$

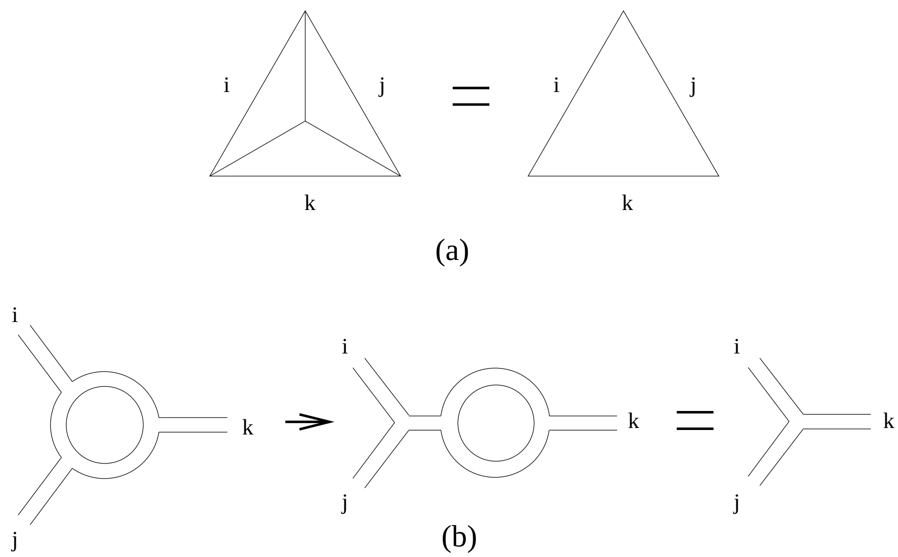


Fig.9. In the limit of $\epsilon \to 0$ both triangles in (a) have zero area. The restriction on the weights can be derived from (b).

where ${B_i}^j$ is defined by

$$B_i^j := \frac{1}{2}\frac{\partial}{\partial \epsilon}\left(C_{ikl}(\epsilon)C^{lkj}(\epsilon)\right)\bigg|_{\epsilon=0}. \tag{5.5}$$

Note that

$$B_{ij} = B_{ji}. \tag{5.6}$$

From (3.5) one sees that to compute the partition function one has simply to calculate $K(\frac{\alpha}{n})^{\frac{n-4g}{2}}$ in the limit $n \to \infty$. Using (5.4) we have

$$\lim_{n\to\infty} K(\frac{\alpha}{n})^{\frac{n-4g}{2}} = \lim_{n\to\infty}(I + \frac{2\alpha}{n}B)^{\frac{n-4g}{2}} = e^{\alpha B}, \tag{5.7}$$

and the partition function is

$$Z(g,\alpha) = Tr\left(e^{\alpha B}H(0)^g\right). \tag{5.8}$$

Equation (5.8) shows that the continuum theory is clearly a deformation of the TFT characterized by A_0. When α goes to zero, $Z(g,0)$ becomes topological.

The operator U also has a well defined continuum limit when restricted to the physical observable. In the limit $n \to \infty$ the algebra A_ϵ becomes A_0. Let σ_1 and σ_2 be the boundary of a cylinder. The observable are given by two loop variables $W(\sigma_1)$ and $W(-\sigma_2)$ belonging to the center of $Z(A_0)$. From (4.15) we have to compute

$$U|_{\text{phy}} = \lim_{n\to\infty} K(\frac{\alpha}{n})^{\frac{n-p_1-p_2-2}{2}}. \tag{5.9}$$

As p_1 and p_2 are of the order $\sqrt{n}$, we get

$$U|_{\text{phy}} = e^{\alpha B} \tag{5.10}$$

where α is the area of the cylinder interpolating between σ_1 and σ_2.

It is clear from the above discussion that the continuum theories are determined by A_0 and an operator $B_i{}^j$. The algebra A_0 defines the topological lattice field theory that one gets in the zero area limit and and $B_i{}^j$ contributes with a non trivial dynamics. Note that $B_i{}^j$ in (5.5) is fixed by the derivative of $C_{ijk}(\epsilon)$ at zero. The global behavior of $C_{ijk}(\epsilon)$ is irrelevant. To classify the possible continuum theories, or universality classes of a LQTFT, one has to determine what are the possible dynamics $B_i{}^j$ that can come from a generic $C_{ijk}(\epsilon)$ via (5.5). As we shall see, for a given A_0, the allowed $B_i{}^j$ are not arbitrary.

Let us call $\Omega(A_0)$ the set of operators $B_i{}^j$ defined by (5.5). Next we show that there is a one to one correspondence between $\Omega(A_0)$ and $Z(A_0)$. First we give a necessary and sufficient condition for a matrix $B_i{}^j$ to be in $\Omega(A_0)$ and then we show the correspondence with $Z(A_0)$.

Consider the matrices $C_m(\epsilon)$, defined by

$$[C_m(\epsilon)]_i{}^j := C_{mi}{}^j(\epsilon) \tag{5.11}$$

One can see from Fig. 10 that $C_m(\epsilon)$ fulfills

$$C_m(\epsilon)K(\epsilon) = K(\epsilon)C_m(\epsilon). \tag{5.12}$$

This equation has to be valid in all orders of $\epsilon = \frac{\alpha}{n}$. It is easy to see that at first order in ϵ, equation (5.12) is equivalent to

$$C_m(0)B = BC_m(0) \quad \text{or} \quad [B, C_m(0)] = 0. \tag{5.13}$$

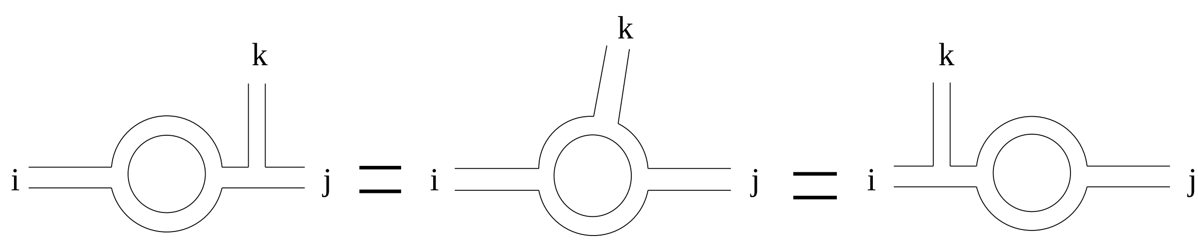


Fig.10. A proof that $C_m(\epsilon)K(\epsilon) = K(\epsilon)C_m(\epsilon)$.

Any operator $B_i{}^j$ coming from (5.5) has to satisfy equations (5.6) and (5.13). Actually this is the only restriction on $B_i{}^j$. Given a topological theory corresponding to the critical point A_0 and an operator $B_i{}^j$ fulfilling (5.6) and (5.13) we can always find at least one $C_{ijk}(\epsilon)$ where $B_i{}^j$ comes from. A simple calculation shows that it is enough to take

$$C_{ijk}(\epsilon) = \left[e^{\epsilon B}\right]_i^l C_{ljk}(0). \tag{5.14}$$

Therefore, $B_i{}^j \in \Omega(A_0)$ if and only if it satisfies equations (5.6) and (5.13).

Now we can set the correspondence between $\Omega(A_0)$ and $Z(A_0)$. Consider the map $\beta : Z(A_0) \to \Omega(A_0)$ that for any $z = z^i \phi_i \in Z(A_0)$ gives a matrix $\beta(z)$ defined by

$$\beta(z)_i{}^j = z^m C_{mi}{}^j(0). \tag{5.15}$$

It is easy to verity that $\beta(z)$ fulfills equations (5.6) and (5.13) and therefore it is indeed an element of $\Omega(A_0)$. The next step is to find the inverse $\beta^{-1} : \Omega(A_0) \to Z(A_0)$. Let $B_i{}^j$ be a matrix in $\Omega(A_0)$ and define

$$\beta^{-1}(B) = C_{ia}{}^a(0) B^{ij} \phi_j. \tag{5.16}$$

We have to make sure that $\beta^{-1}(B) \in Z(A_0)$. For that consider the sequence of moves in Fig. 11 (a). Comparing the first and the last graph we have

$$C_{ij}{}^k(0) B^{ib} C_{ba}{}^a(0) = C_{ji}{}^k(0) B^{ib} C_{ba}{}^a(0).$$

Therefore $\beta^{-1}(B)\phi_j = \phi_j \beta^{-1}(B)$.

It is very simple to verify that $\beta^{-1} \circ \beta$ is the identity map in $Z(A_0)$. Given an element $z^m \phi_m \in Z(A_0)$ we will have

$$\left[(\beta^{-1} \circ \beta)(z^m \phi_m)\right] = \beta^{-1}(z^m C_m) = C_{ia}{}^a(0) z^m C_m{}^{il}(0)\phi_l = z^m \phi_m. \tag{5.17}$$

We can also show that $\beta \circ \beta^{-1}$ is the identity map in $\Omega(A_0)$. For that consider Fig. 11 (b). The matrix $B \in \Omega(A_0)$ is displayed as a box. It follows from its commutation with $C_m(0)$ that we can attach the box on any side of the $C_{ij}{}^k$, hence the first step of the figure. Fig. 11 (b) shows that $[(\beta \circ \beta^{-1})(B)]_i{}^j = B_i{}^j$, and therefore $\beta \circ \beta^{-1}$ is the identity map.

In conclusion, the map β defined in (5.15) is a bijection. In other words all matrices of $\Omega(A_0)$ are of the form $z^k C_m(0)$ for some $z = z^k \phi_k \in Z(A_0)$.

6 Example

We will now consider an example of quasi-topological theory in the continuum limit. We will study the case where the zero area topological theory is derived from a group algebra A_0.

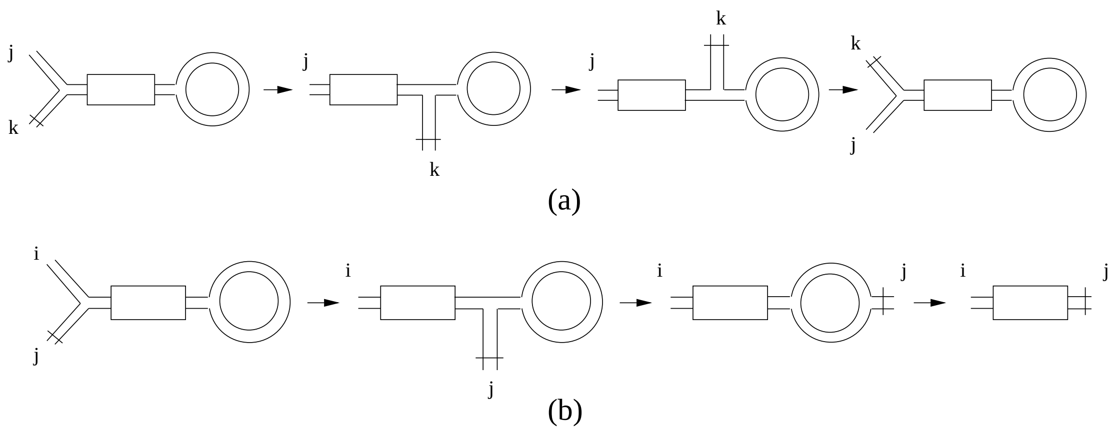


Fig.11. Figure (a) shows that the image of β is in the center. Figure (b) shows that $\beta \circ \beta^{-1}$ is the identity map. The crossed strips mean that the corresponding index is raised.

Given a group G, we can construct a group algebra A_0 over the complex numbers in the usual way:

$$\mathbf{C}[G] = \bigoplus_{g \in G} \mathbf{C}\phi_g,$$

with the algebra product inherited from the group, i.e., $\phi_x \phi_y \equiv \phi_{xy}$.

We can then calculate ${C_{ij}}^k(0)$:

$${C_{ij}}^k(0) = \delta(ij, k) \ , \quad g_{ij} = \delta(i, j^{-1}) \ , \quad C_{ijk}(0) = \delta(ijk, \mathbb{1}),$$

where $\mathbb{1}$ is the identity element in the group.

The next step is to determine $\Omega(A_0)$, or the set of all quasi-topological deformation in the continuum. From Section 5, we know that $\Omega(A_0)$ is given by all B_{ij} fulfilling (5.6) and (5.13). If we denote $B_i^{\ j}$ by $B(i, j)$, equation (5.13) reads

$$\sum_{l \in G} \delta(ij, l) B(l, k) = \sum_{l \in G} \delta(il, k) B(j, l), \tag{6.1}$$

or in other words $B(ij, k) = B(j, i^{-1}k)$. Therefore

$$B(i, j) = B(j^{-1}i, \mathbb{1}). \tag{6.2}$$

Using the fact that $B_{ij} = B_{ji}$ together with (6.2) it is easy to show that

$$B(i, j) = B(ij^{-1}, \mathbb{1}). \tag{6.3}$$

Equations (6.2) and (6.3) makes clear that the operator $B(i, j)$ is determined by a single function $h : G \to \mathbf{C}$ defined as

$$h(k) = B(k, \mathbb{1}). \tag{6.4}$$

Furthermore such function satisfies $h(ij) = h(ji)$. Therefore

$$h(kik^{-1}) = h(i) \tag{6.5}$$

and $h(k)$ is a class function, i.e., it depends only on the conjugacy classes. As a class function, $h(k)$ can be expanded on the characters χ_R of the group. Therefore we can write the operator $B(i,j) = h(ij^{-1})$ as

$$B(i,j) = \sum_R B_R d_R \chi_R(ij^{-1}), \tag{6.6}$$

where the sum runs over all irreducible representations of G. The complex constants B_R are arbitrary in order to span all possible operators $B(i,j)$.

As we have seen on Section 5 that $\Omega(A_0)$ is in one to one correspondence with the center $Z(A_0)$. On the other hand, equation (6.6) shows that $\Omega(A_0)$ is spanned by the class functions. Both results are actually equivalent since there is a one to one correspondence between the set of all class functions and the center of the group algebra.

Each choice of coefficients B_R in (6.6) gives us a different quantum field theory in the continuum. It is interesting to go a little further and try to identify what could be the Lagrangian formulation of such field theories. This problem can be solved by calculating the partition function for a triangle Δ of area α in the continuum limit. For this let us subdivide Δ in to a triangulation such that the external edges of Δ are not subdivided. In other words, the corresponding graph has only three external legs. One can get such triangulation starting from the one of a cylinder, by closing one of the boundaries. This correspond to a particular case of the graph on Fig. 7, where there are only 3 external legs colored by i, j, k on one side an no external legs on the other side. The resulting graph can be further simplified by applying a sequence of flip moves and it is equivalent to the one on Fig. 12. The evaluation of the chain of operators $K_i{}^j(\epsilon)$ gives the exponential of

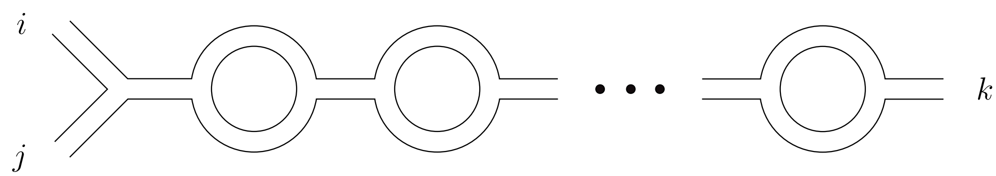


Fig.12. The standard graph for a triangle with finite area.

B. The result

$$(e^{\alpha B})_i{}^j = \sum_R e^{\alpha B_R} d_R \chi_R(ij^{-1}) \tag{6.7}$$

can be easily computed using using (6.6) and the orthogonality of the characters. From Fig. 12, follows that the partition function for the triangle at this limit is:

$$Z(i,j,k,A) = C^0_{ijl} (e^{\alpha B})_k{}^l = \sum_R e^{\alpha B_R} d_R \chi_R(ijk). \tag{6.8}$$

Consider a particular case of (6.8) where G is a Lie group and B_R is equal to the quadratic Casimir operator $C_2(R)$. The reader will recognize the heat kernel action for a triangular plaquette that gives Yang-Mills theory in the continuum. This fact allow us to identify the Lagrangian formulation for this particular quasi-topological theory as being 2d Yang-Mills. If B_R is not equal to $C_2(R)$ the identification problem becomes less clear. For such choices of the coefficients B_R the corresponding theories are not YM_2 but they are still deformations of the same topological theory with the area being the deformation parameter. All one has to do is to find a Lagrangian field theory that gives (6.8) as the partition function for a triangle. Such field theories exist and are called generalized YM_2. We refer to [8] for the relevant computations and further details.

7 Concluding Remarks

Two dimensional lattice quasi-topological field theories are "less trivial" than topological models in the sense that they have nontrivial dynamics. On the other hand they have less symmetry than the topological models. In the continuum limit the invariance under the whole diffeomorphism group is broken down to the area preserving diffeomorphisms. Nevertheless the residual symmetry is still enough to make the model soluble as it allows to reduce a two dimensional problem to a one dimensional computation. If the link variables assume values in a finite dimensional set, the partition function can be exactly computed.

The set of Boltzmann weight ${C_{ij}}^k(\epsilon)$ and the gluing operator g^{ij} give a one parameter family of associative algebras A_ϵ together with a bilinear form. We refer to this data as a pair (A_ϵ, g^{ij}). The scaling limit $\epsilon \to 0$, $n = \frac{\alpha}{\epsilon}$ is well defined whenever ${C_{ij}}^k(0)$ and g^{ij}, define a lattice topological field theory. At $\epsilon = 0$ the topological symmetry is restored and the theory becomes invariant by subdivision. The continuum theory is not topological and the partition function depends also on the total area of the surface. As it is required by consistence, the theories become topological in the zero area limit. This is what is meant by a quasi-topological field theory, the prototype being YM_2 [7]. We have seen that a single topological theory given by A_0 can be the zero area limit of more that one continuum quasi-topological theory. The particular continuum limit will depend on how ${C_{ij}}^k(\epsilon)$ approaches the critical point $\epsilon = 0$. This is measured by the operator ${B_i}^j$ defined in (5.5). We have seen that the set of all quasi-topological theories associated with A_0 is in one to one correspondence to the center $Z(A_0)$ of the semi-simple algebra A_0.

It is not clear which quasi-topological theories in the continuum can be described by means of a Lagrangian. The Lagrangian approach is certainly possible in the case of YM_2 and generalized YM_2. It would be very interesting to find other examples of such Lagrangian theories. The simplest solution is to look for the analog of a Schwarz type topological field theory, or in other words, actions that are invariant under area preserving diffeomorphisms. If there are no anomalies, the zero area limit should be a Schwarz type

topological field theory. Volume preserving theories have been considered in [10]. However this may not be generic enough and one may need to find quasi-topological theories that in the zero area limit reduces to Witten's type topological field theories . This possibility is presently under investigation and results will be reported elsewhere.

A Appendix

We will now present a proof that all triangulations of a given genus $g > 0$ surface consisting of n triangles can be connected by a sequence of flip moves. This will be done by an argument somewhat similar to the one of [9]. The idea is to reduce any triangulation to a special one which we will refer as standard triangulation. It consists of several bubble-like structures composed of two triangles, as well as some double-handled structures composed of four triangles. The double handled structures give information about the genus. The dual graph of a standard triangulation and the two basic building blocks are shown in Fig. 4.

It is well known that any surface of genus g can be represented as a $4g$-sided polygon with its sides identified suitably [11]. Let us enumerate the sides of the polygon as $a_1, b_1, a_1^{-1}, b_1^{-1}, ..., a_g, b_g, a_g^{-1}, b_g^{-1}$. The g genus surface is recovered by gluing the sides a_i and b_i with a_i^{-1} and b_i^{-1} respectively.

There is a natural distinction among the triangles of the triangulated $4g$-sided polygon. We will call the triangles external or internal according with whether they share or not an edge with the boundary of the polygon. Such a distinction disappears after the sides of the polygon are identified. In this way many different triangulations of the polygon can give the same triangulation of the surface. The minimum number of external triangles is clearly $4g$.

As a first step towards the standard triangulation we will show that no matter how complicated is the triangulation of the surface, we can always reduce the number of external triangles per side of the polygon by one and therefore reduce it to the minimal number. In other words, it is enough to consider triangulated polygons with only $4g$ external triangles. Suppose that there is a side of the polygon consisting of l links. Consider any two consecutive links, say $\overline{AB}$ and $\overline{BC}$, and their respective triangles. Observe that one can always perform a sequence of flip moves in order to make these two triangles share an edge. Now we follow the steps shown in Fig. 13. We can flip the common edge and as a result the two consecutive links now belong to the same triangle ABC. To proceed, remember that this polygon is in fact a genus g surface, and therefore the side containing $\overline{ABC}$ is identified with another side. Clearly the notion of which sequence of links makes the identified sides is somewhat arbitrary. The links defined as the side of the polygon could be in fact any edge nearby. In particular, we can replace $\overline{ABC}$ by $\overline{AC}$. In this way we arrive at the last picture of Fig. 13. The number of links in a given side of the polygon

has been reduced by one. We get a triangulated polygon with $4g$ external triangles by iterating the process.

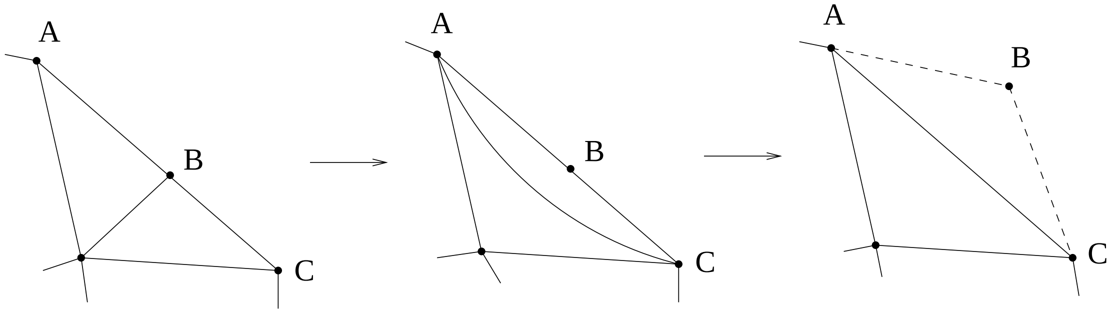


Fig.13. How we can decrease the number of external triangles. From left to right: in the first figure we have two triangles, and the external edges are $\overline{ABC}$, in the second we made a flip move, in the third we redefined our external edge as $\overline{AC}$. The dashed line means that B has been sent "to the other side".

The dual graphs of triangulated polygons are planar. Therefore we can simplify the pictures by using single lines instead of double lines when drawing them. The $4g$ external triangles will be represented by $4g$ external legs in the dual graph numbered by $a_1, b_1, a_1^{-1}, b_1^{-1}, ..., a_g, b_g, a_g^{-1}, b_g^{-1}$. It is not difficult to see what is the general structure of the dual graphs. After considering some examples, like the one in Fig. 2, one realizes that the graphs consist of a big external with the $4g$ external legs attached to it plus several internal loops all interconnected. We now follow [9] to arrive at the standard triangulation for the $4g$-sided polygon.

One should remember that the action of the flip move is simple the sliding of one edge over another as shown on Fig. 14.

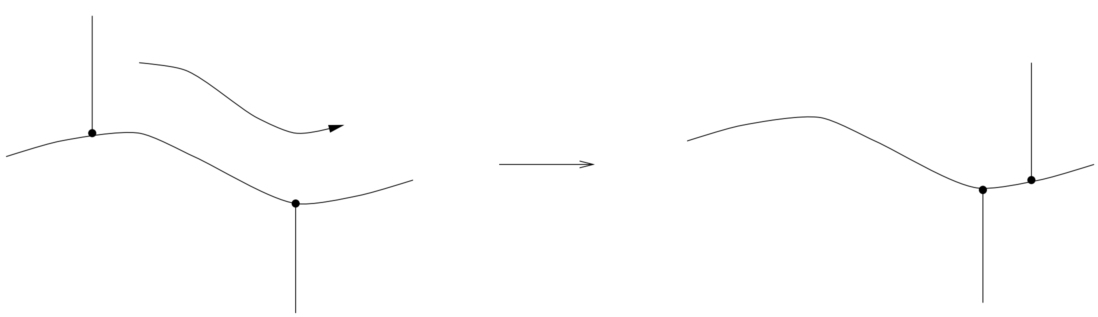


Fig.14. The action of the flip move on dual graphs. As the graph is planar, we can represent it by single lines. The figure shows the flip move as the sliding of lines one over another.

Consider now the internal loops. By sliding one line over another, we can arrange all lines connecting the several loops in such way that a given loop is linked to at most two different loops. Fig. 15 shows how to disentangle compounds of this type, forming "pins" in the process. We are left then with chain-like structures of loops linked by some various

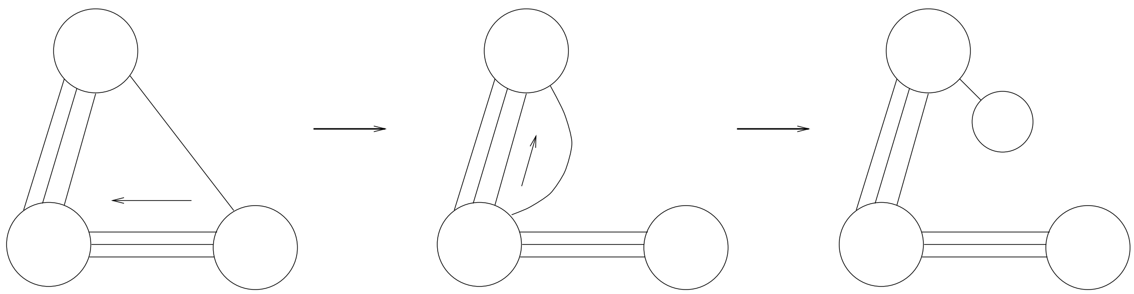


Fig.15. How to disentangle interconnected loops. This results in a number of "pins" attached to any of them.

numbers of lines. Fig. 16 shows how to reduce the number of lines to just one line, again forming pins. We have by now some internal loops, with some pins attached, connected

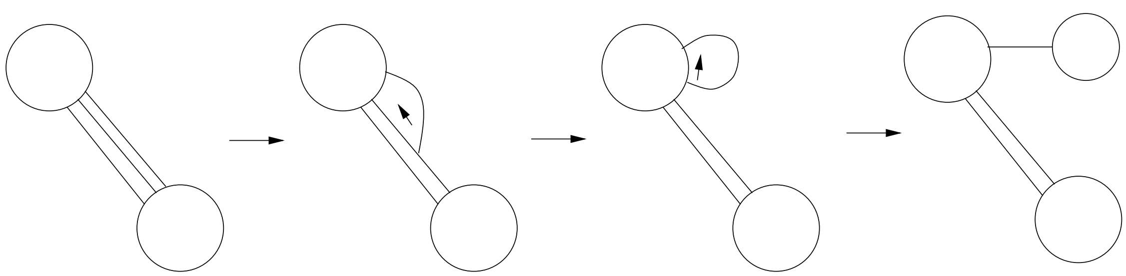


Fig.16. How to reduce the number of lines connecting two loops creating "pins".

by just one line to at most two different loops, and this compound connected by just one line to a greater external loop. The pins can be carried one by one to the last internal loop – the only loop that is connected to just one different loop – and thus becoming the last one. Repeating this process, we will eventually reach a triangulation with a chain of loops linked by only one line to each other. One end of the chain will be linked from inside with the large external loop. After gluing the external legs labeled a_i and b_i with a_i^{-1} and b_i^{-1} one gets graph equivalent to the one on Fig. 4 (a).

One may notice that, although all we did was for a surface with genus greater than zero, we could also extend the argument for genus zero. However, this is exactly what is done in [9] and we shall not repeat the argument here.

Acknowledgments
This work was supported by CNPq.

References

[1] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, New York, 1982)

[2] S. M. Bhattacharjee and A. Khare, *Fifty Years of the Exact Solution of the Two-dimensional Ising Model by Onsager*, preprint IOP-BBSR/95, `cond-mat/9511003`.

[3] J.F. Wheater, *Phys. Lett.* **B223** (1989), 551; T. Jonsson, Phys. Lett. **B265** (1991), 141; C. Bachas and P.M.M. Petropoulos, *Commun. Math. Phys.* **152** (1993), 191.

[4] M. Fukuma, S. Hosono, H. Kawai, *Commun. Math. Phys.* **161** (1994), 157.

[5] S-W Chung, M. Fukuma, A. Shapere, *Int. J. Mod. Phys.* **A9** (1994) 1305.

[6] D. Birmingham and M. Rakowski, *Commun. Math. Phys.* **173** (1995) 135, `hep-th/9405108`; *Int.J.Mod.Phys.* **A10** (1995), `hep-th/9303110`.

[7] E. Witten, *Commun. Math. Phys.* **141** (1991), 153; M. Blau, G. Thompson, *Int. J. Mod. Phys.* **A7** (1992) 3781.

[8] E. Witten, *J. Geom. and Physics* **9** (1992) 303; M.R. Douglas, K. Li and M. Staudacher, *Nucl. Phys.* **B240** (1994) 118; O. Ganor, J. Sonnenschein and S. Yankielowicz, *Nucl. Phys.* **B434** (1995) 139; M. Khorrami and M. Alimohammadi, `hep-th/9612018`.

[9] D. V. Boulatov, V.A. Kazakov, I.K. Kostov and A.A. Migdal, *Nucl. Phys.* **B275**, (1986), 641.

[10] R. Brooks, *Nucl. Phys.* **B423** (1994) 197, `hep-th/9310007`.

[11] Se for example J. Stillwell, *Classical Topology and Combinatorial Group Theory*, Springer-Verlag, New York, 1980.